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5524004 Martin Co., Baltimore, Md. RIAS Div.

Research Institute for Advanced Studies (RIAS)
7212 Bellona Avenue
Baltimore 12, Maryland

Generalized recurrence in dynamical systems

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Joseph Auslander*

1. Introduction. In this paper, a notion of recurrence in dynamical systems is introduced which is defined using continuous real valued functions on the phase space. Our basic idea is as follows. Consider the class of continuous real valued functions which are non-increasing along every orbit. Then we single out those orbits along which all such functions are constant. This set, which includes the periodic, recurrent, and non-wandering points, is called the generalized recurrent set. Its elementary properties are studied in section 1, and it is shown (Theorem 2) that a single suitably chosen function reflects the "recursive" behavior of the dynamical system. By means of prolongations, an intrinsic characterization of the generalized recurrent set is given in section 3. This depends on a purely topological result involving a closed quasi order (Theorem 4) which is apparently new and may be of independent interest. In section 4, the connection of the generalized recurrent set with asymptotic and absolute stability of a compact invariant set is discussed. The condition that the dynamical system be free of generalized recurrent orbits is shown to lie between parallelizability and complete instability (Theorem 6).

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In what follows, X denotes a locally compact separable metric space, with metric d, and E denotes the real line. A *dynamical system* or *continuous flow* $\mathcal{F}$ on X is a continuous map $\pi : X \times E \to X$ satisfying

a) $\pi(x, 0) = x \qquad (x \in X)$

b) $\pi(\pi(x, t_1), t_2) = \pi(x, t, + t_2), \quad (x \in X, \ t_1, t_2 \in E)$.

From now on, we write xt in place of $\pi(x, t)$; axiom b) then reads $(xt_1)t_2 = x(t_1 + t_2)$.

If $x \in X$, the *orbit* of x is the set $\gamma(x) = \{xt \mid t \in E\}$. The *positive* and *negative semi-orbits of* x are the sets $\gamma^+(x) = \{xt \mid t \geq 0\}$, and $\gamma^-(x) = \{xt \mid t \leq 0\}$ respectively.

We briefly recall some of the notions of recurrence which have occurred in the study of dynamical systems, [3], [5]. They all express the idea of a point returning to itself, in some sense, for arbitrarily large time. The simplest is a periodic point, that is, a point x such that $x\tau = x$, for some $\tau > 0$. A somewhat weaker notion is that of a *recurrent* or *Poisson stable* point. The point x is called (positively) recurrent if, for every neighborhood U of x, and every $T > 0$, there is a $t > T$ such that $xt \in U$. Clearly x is recurrent if and only if there exists a sequence $\{t_n\}$ of real numbers with $t_n \to +\infty$ and $xt_n \to x$. Still weaker is a *non-wandering* point -- that is, a point x such that for every neighborhood U of x and every $T > 0$, $U \cap Ut \neq \emptyset$, for some $t > T$. Obviously, the set of non-wandering points contains the set of recurrent points, which in turn contains the periodic points; well known examples show that these inclusions may be proper.

2. Generalized recurrence. Let $\mathcal{V}$ denote the class of continuous real valued functions f on X such that $f(xt) \leqq f(x)$, for all $x \in X$ and all $t > 0$. Let z be a non-wandering point. If $f \in \mathcal{V}$, then f is constant on $\gamma(z)$. To show this, let $\tau > 0$. Since z is non-wandering, there are sequences $z_n \to z$ and $t_n \to \infty$ such that $z_n t_n \to z$. For n sufficiently large $f(z_n t_n) \leqq f(z_n \tau)$, and by continuity of f, $f(z) \leqq f(z\tau)$. Since $f \in \mathcal{V}$, $f(z\tau) \leqq f(z)$, and therefore $f(z\tau) = f(z)$. If $\tau < 0$, apply the same argument to $z\tau$ and $z = (z\tau)(-\tau)$.

We define the set $\mathcal{R}$ to be the set of points $x \in X$ such that $f(xt) = f(x)$, for all $f \in \mathcal{V}$, and all $t \geqq 0$. $\mathcal{R}$ will be called the generalized recurrent set or sometimes the recurrent set. The above discussion shows that $\mathcal{R}$ includes the non-wandering points, and therefore the (ordinary) recurrent points and the periodic points.

Theorem 1. *$\mathcal{R}$ is a closed, positively and negatively invariant set.*

Proof. $\mathcal{R}$ is obviously closed. Let $x \in \mathcal{R}$, and let $\tau > 0$. Let $f \in \mathcal{V}$. If $t > 0$, then $f((x\tau)t) = f(x(\tau + t)) = f(x) = f(x\tau)$. Suppose $x\tau \notin \mathcal{R}$, for some $\tau < 0$. Then there is a $g \in \mathcal{V}$ and a $t_o > 0$ such that $g((x\tau)t_o) < g(x\tau)$. Let $f \in \mathcal{V}$ be defined by $f(z) = g(z\tau)$ $(z \in X)$. Then $f(xt_o) = g((xt_o)\tau) = g((x\tau)t_o) < g(x\tau) = f(x)$. This contradicts $x \in \mathcal{R}$.

Since $\mathcal{R}$ is invariant, it is meaningful to speak of recurrent orbits.

If $f \in \mathcal{V}$, then so are $\arctan f$ and $cf + d$, where c and d are real numbers with $c \geqq 0$. This remark and Theorem 1 yield

Lemma 1. *Let a and b be real numbers with $a < b$. Let $\mathcal{V}_{a,b} = \{f \in \mathcal{V} \mid a \leqq f(x) \leqq b$, for all $x \in X\}$. Then $x \in \mathcal{R}$ if and only if $f(xt) = f(x)$ for all $f \in \mathcal{V}_{a,b}$ and all real t.*

According to the definition, a point x is in $\mathcal{R}$ only if all $f \in \mathcal{V}$ are constant on its orbit. The next theorem shows that there is a "universal" f for this purpose.

Theorem 2. *There is an* f *in* $\mathcal{V}$ *with the following properties:*

(i) $x \in \mathcal{R}$ *if and only if* f *is constant on* $\gamma(x)$.

(ii) *If* $x \notin \mathcal{R}$, *and* $t > 0$, *then* $f(xt) < f(x)$.

Proof. Let $C(X)$ denote the continuous real valued functions on X, provided with the topology of uniform convergence on compact sets. Then $C(X)$ is a separable metric space and so is $\mathcal{V}^* = \mathcal{V}_{-1,1}$. Let $\{f_k\}$ $(k = 1,2, \ldots)$ be a countable dense set in $\mathcal{V}^*$. Then $x \in \mathcal{R}$ if and only if $f_k(xt) = f_k(x)$ for $k = 1,2, \ldots,$ and all $t \in E$. Now, let $f^* = \sum_{k=1}^{\infty} \frac{1}{2^k} f_k$. Since $|f_k(x)| \leqq 1$, it follows that f^* is continuous and in $\mathcal{V}^*$. If $f^*(xt) = f^*(x)$, for all $t > 0$, then $f_k(xt) = f_k(x)$ for $k = 1,2, \ldots,$ and $x \in \mathcal{R}$. If $x \notin \mathcal{R}$, there is a sequence $\{t_n\}$ in E with $t_n \to +\infty$ such that $f^*(x) > f^*(xt_1) > f^*(xt_2) > \ldots$. Define $f(x) = \int_0^\infty \alpha(s) f^*(xs)ds$, when α is a positive strictly decreasing function such that $\int_0^\infty \alpha(s)ds < 1$. It is easily verified that $f \in \mathcal{V}^*$ and has the required properties.

3. Another characterization of $\mathcal{R}$. We wish to characterize the set intrinsically -- that is, solely in terms of the dynamical system $\mathcal{F}$. We do this by means of the theory of prolongations, developed by Ura, [6], [7], and Seibert and the author, [2]. We now review those concepts and results from this theory which we will need in this section. The reader is referred to the papers cited above for details.

Let P_o be a map from X to 2^X, the set of all subsets of X. We define two new maps, $\mathcal{D} P_o$ and $\mathcal{S} P_o$, from X to 2^X by $\mathcal{D} P_o(x) = \bigcap_{U \in \mathcal{N}(x)} \overline{P_o(U)}$ (where $\mathcal{N}(x)$ denotes the neighborhood filter of x), and $\mathcal{S} P_o(x) = \bigcup_{n=1,2,\dots} P_o^n(x)$. It is easy to see that $y \in \mathcal{D} P_o(x)$ if and only if there are sequences $\{x_n\}$ and $\{y_n\}$ in X with $y_n \in P_o(x_n)$ such that $x_n \to x$ and $y_n \to y$. Also $y \in \mathcal{S} P_o(x)$ if and only if there are points $x_o = x$, x_1, ..., $x_n = y$, with $x_i \in P_o(x_{i-1})$ $(i = 1,2, \dots, n)$. The maps $\mathcal{S}$ and $\mathcal{D}$ may be regarded as operators on the set of maps from X to 2^X; $\mathcal{D}$ is a "closure" operator and $\mathcal{S}$ is a "transitizing" operator.

We define a family of maps P_α from X to 2^X inductively. Let $P_1 = \mathcal{D}\mathcal{S} P_o$. Now, let α be an ordinal number, and suppose P_β has been defined for $\beta < \alpha$. We define $P_\alpha = \mathcal{D}(\bigcup_{\beta < \alpha} \mathcal{S} P_\beta)$. (By $\bigcup Q_\beta$ we mean the map Q given by $Q(x) = \bigcup Q_\beta(x)$). If α is a successor ordinal, this definition reduces to $P_\alpha = \mathcal{D}\mathcal{S} P_{\alpha-1}$. Clearly, if $\beta < \alpha$, then $P_\beta(x) \subset P_\alpha(x)$. If α is sufficiently large (equal to the first uncountable ordinal ζ) it may be shown that $\mathcal{D} P_\alpha = \mathcal{S} P_\alpha = P_\alpha$, and hence that $P_\lambda = P_\alpha$ for all $\lambda \geq \alpha$. We write $P^* = \bigcup P_\alpha$, which is equal to P_ζ.

This precedure is applied to two maps which are connected with the dynamical system $\mathcal{F}$. We let $D_o(x) = \gamma^+(x)$, the positive semi-orbit of x. As above, define $D_1 = \mathcal{D}\mathcal{S} D_o = \mathcal{D} D_o$ (since $\mathcal{S} D_o = D_o$ already), $D_2 = \mathcal{D}\mathcal{S} D_1$, ..., $D_\alpha = \mathcal{D}(\bigcup_{\beta < \alpha} \mathcal{S} D_\beta)$, and $D^* = \bigcup D_\alpha$. The maps D_α are the <u>prolongations</u> of Ura. The other map under consideration, denoted by J_1, is defined by $y \in J_1(x)$ if and only if there are sequences $\{x_n\}$, and $\{y_n\}$ in X, and $\{t_n\}$ in E, with $t_n \to +\infty$ such that $x_n \to x$, $y_n \to y$

and $y_n = x_n t_n$. Then define $J_2 = \mathcal{D}\mathcal{S}J_1, \ldots, J_\alpha = \mathcal{D}(\bigcup_{\beta<\alpha} \mathcal{S}J_\beta)$, and $J^* = \cup J_\alpha$.

It is clear that y is in $D_1(x)$ if and only if there are sequences $\{x_n\}$, $\{y_n\}$ in X, and $t_n \geq 0$ with $x_n \to x$, $y_n = x_n t_n$, and $y_n \to y$. This tells us that $J_1(x) \subset D_1(x)$; it follows by an easy induction that $J_\alpha(x) \subset D_\alpha(x)$ for all $\alpha \geq 1$.

We define $\mathcal{R}_\alpha$ to be the set of $x \in X$ such that $x \in J_\alpha(x)$. Let $\mathcal{R}^* = \cup \mathcal{R}_\alpha$. Our alternate characterization of $\mathcal{R}$ is given in the next theorem.

<u>Theorem 3</u>. $\mathcal{R} = \mathcal{R}^*$. <u>That is,</u> $x \in \mathcal{R}$ <u>if and only if</u> $x \in J_\alpha(x)$, <u>for some</u> α.

The proof of Theorem 3 is preceded by a sequence of lemmas.

<u>Lemma 2</u>. $J_\alpha(x)$ <u>is a positively and negatively invariant set.</u>

Proof. Let $y \in J_1(x)$, and $t \in E$. Then there are sequences $\{x_n\}$ in X, $\{t_n\}$ in E, with $x_n \to x$, $t_n \to \infty$ such that $x_n t_n \to y$. Then $t_n + t \to \infty$, and $x_n(t_n + t) \to yt$, so $yt \in J_1(x)$. Suppose the lemma is true for all $\beta < \alpha$. Let $y \in J_\alpha(x)$ and $t \in E$. Let $x_n \to x$, $y_n \to y$, where $y_n \in J_{\beta_n}^{k_n}(x_n)$ ($\beta_n < \alpha$, k_n a positive integer). By the induction hypothesis $y_n t \in J_{\beta_n}^{k_n}(x_n)$, and $y_n t \to yt \in J_\alpha(x)$.

<u>Lemma 3</u>. $J_\alpha(xt) = J_\alpha(x)t = J_\alpha(x)$, <u>for all</u> $t \in E$.

Proof. By Lemma 2, $J_\alpha(x)t = J_\alpha(x)$. It is an immediate consequence of the definition that $J_1(xt) = J_1(x)t$. Suppose $J_\beta(xt) = J_\beta(x)t$, for all $\beta < \alpha$, and let $y \in J_\alpha(xt)$. Let $y_n \in J_{\beta_n}^{k_n}(x_n)$ (where $\beta_n < \alpha$, and $k_n \geq 1$)

such that $x_{k_n} \to xt$ and $y_{k_n} \to y$. Now $x_n(-t) \to x$, and $y_n(-t) \in J_{\beta_n}^{k_n}(x_n)(-t) = J_{\beta_n}^{k_n}(x_n)$, by the induction assumption. Then $y_n(-t) \to y(-t)$, so $y(-t) \in J_\alpha(x)$, and $y \in J_\alpha(x)t$. Hence $J_\alpha(xt) \subset J_\alpha(x)t$. Now $J_\alpha(x)t = J_\alpha(xt(-t))t \subset J_\alpha(xt)(-t)(t) = J_\alpha(xt)$.

It follows from Lemma 3 that $\mathcal{R}^*$ is a positively and negatively invariant set.

<u>Lemma 4.</u> $D_\alpha(x) = J_\alpha(x) \cup \gamma^+(x)$.

Proof. For $\alpha = 1$ the lemma is easy. Suppose it is true for all $\beta < \alpha$. Observe that if $y' \in D_\beta^k(x')$, then, by Lemma 3, $y' \in \gamma^+(x')$, or $y' \in J_\beta^m(x')$, where $m \leq k$. Now, if $y \in D_\alpha(x)$, let $x_n \to x$, $y_n \to y$, with $y_n \in D_{\beta_n}^{k_n}(x_n)$ $(\beta_n < \alpha,\ k_n \geq 1)$. If $y_n \in J_{\beta_n}^{\ell_n}(x_n)$ $(\ell_n \leq k_n)$, for infinitely many n, $y \in J_\alpha(x)$. If $y_n \in \gamma^+(x_n)$, for infinitely many n, then $y \in D_1(x) = \gamma^+(x) \cup J_1(x) \subset \gamma^+(x) \cup J_\alpha(x)$. The proof is completed.

<u>Lemma 5. The following are equivalent:</u>

(i) $x \in \mathcal{R}_\alpha$

(ii) $\gamma^-(x) \subset D_\alpha(x)$

(iii) $x\tau \in D_\alpha(x)$, for some $\tau < 0$

(iv) $D_\alpha(x) = J_\alpha(x)$.

Proof. (i) ==> (ii) is an immediate consequence of Lemma 3, and (ii) ==> (iii) is clear. Suppose (iii) holds. To show (iv), it is sufficient, by Lemma 4, to show $\gamma^+(x) \subset J_\alpha(x)$. Let $\tau < 0$ such that $x\tau \in D_\alpha(x) = \gamma^+(x) \cup J_\alpha(x)$. If $x\tau \in \gamma^+(x)$, then x is periodic, and

$x \in J_\alpha(x)$. Otherwise $x\tau \in J_\alpha(x)$. Let $t \geq \tau$. Then $(x\tau)t \in J_\alpha(x)t = J_\alpha(x)$, and $\gamma^+(x) \subset J_\alpha(x)$. Since $x \in D_\alpha(x)$ always, (iv) ==> (i) is obvious.

Lemma 6. If $f \in \mathcal{V}$, and $y \in D_\alpha(x)$, then $f(y) \leq f(x)$.

This is proved by a trivial induction.

Now we can prove that $\mathcal{R}^* \subset \mathcal{R}$. Let $x \in \mathcal{R}_\alpha$, and let $f \in \mathcal{V}$. Suppose $t > 0$. Then $x \in J_\alpha(x) = J_\alpha(xt)$, by Lemma 3. Since $J_\alpha(xt) \subset D_\alpha(xt)$, Lemma 6 implies $f(x) \leq f(xt)$. Since always $f(xt) \leq f(x)$, $x \in \mathcal{R}$.

To prove that $\mathcal{R} \subset \mathcal{R}^*$, we define a relation $\prec$ on X by $y \prec x$ if and only if $y \in D^*(x)$. Then $\prec$ is a closed quasi order on X. (By a quasi order we mean a reflexive, transitive, but not necessarily anti symmetric relation.) Observe that $xt \prec x$, whenever $x \in X$, and $t > 0$. If $x \prec y$ but not $y \prec x$, we write $x \prec\prec y$. We first show: if $x \notin \mathcal{R}^*$, and $t > 0$, then $xt \prec\prec x$. We know that $xt \prec x$. If $x \prec xt$, then $x \in D^*(xt)$, and $xt \in \mathcal{R}^*$, by Lemma 5. Then $x \in \mathcal{R}^*(-t) = \mathcal{R}^*$ and this is a contradiction.

Theorem 3 now is a consequence of the following purely topological theorem which has nothing to do with dynamical systems.

Theorem 4. Let X be a separable locally compact metric space and let $\prec$ be a closed quasi-order on X. Let x and y in X such that $x \prec y$ does not hold. Then there is an f in C(X) such that (i) if $z \prec z'$, then $f(z) \leq f(z')$, (ii) $f(y) < f(x)$.

This theorem will be proved in section 5.

4. Recurrence and stability. In [2], the prolongations D_α were employed to study the stability properties of a compact positively invariant set M. We say that M is stable of order α or α-stable if $D_\alpha(M) = M$. It is easy to see that stability of order one is just Liapunov stability. If M is α-stable for every ordinal number α, then M is said to be absolutely stable.

Absolute stability of M is equivalent to the existence of a continuous Liapunov function for M (that is, a continuous non-negative function V defined in a positively invariant neighborhood U of M such that $V(x) = 0$ if and only if $x \in M$, and such that $V(xt) \leqq V(x)$ for $x \in U$, and $t > 0$) ([2], Theorem 3).

Now, while evidently absolute stability is a rather strong condition, simple examples show that it is weaker than asymptotic stability (that is, Liapunov stability as well as the existence of a neighborhood U of M such that $xt \to M$, as $t \to \infty$, for all $x \in U$). We shall show that the difference between absolute stability and asymptotic stability is the existence of generalized recurrent orbits arbitrarily close to M.

Let M be an absolutely stable compact positively invariant set, and let V be a continuous Liapunov function for M, defined in a neighborhood U of M. Then there exists $\epsilon > 0$ such that $U^* = \{x | V(x) \leqq \epsilon\}$ is a compact subset of the interior of U.

We may suppose $\epsilon = 1$. Let f denote a "universal" function in $\mathcal{V}$, as constructed in Theorem 2, such that $f(x) > 0$, for all $x \in X$. Define

$$V'(x) = \begin{cases} V(x)f(x) & x \in U^* \\ f(x) & x \in X - U^* \end{cases}$$

Then V' is a continuous Liapunov function for M, which is defined on all of X, so $V' \in \mathcal{V}$. Moreover, it is clear that if $x \in \mathcal{R}$, $V'(xt) = V'(x)$, and if $x \in X - \mathcal{R} - M$, $V'(xt) < V'(x)$, for all $t > 0$.

Lemma 7. *Let M be absolutely stable, and let U be a compact positively invariant neighborhood of M. Let $M^* = (M \cup \mathcal{R}) \cap U$. Then, for all $x \in U$, $xt \to M^*$ as $t \to \infty$.*

Proof. First observe that M^* is positively invariant. Let V' be the Liapunov function constructed above, and let $x \in U - M^*$. Let $y \in \Omega(x)$, the omega limit set of x. Then there is a sequence $\{t_n\}$ in E with $t_n \to \infty$ such that $xt_n \to y$. Now $\lim_{t \to \infty} V'(xt) = \lambda$ exists, and obviously $V'(y) = \lambda$. Moreover, if $\tau \in E$, $t_n + \tau \to \infty$, so $V'(x(t_n + \tau)) \to V'(y\tau) = \lambda$. Therefore $y \in U \cap \mathcal{R} \subset M^*$.

The method of proof of this lemma is due to LaSalle, ([4], Theorem 1.)

Theorem 5. *Let M be absolutely stable. Then M is asymptotically stable if and only if there is a neighborhood U of M such that $(U - M) \cap \mathcal{R} = \Phi$.*

Proof. If M is asymptotically stable, there exists a neighborhood W of M and a continuous Liapunov function V for M such that $V(xt) < V(x)$, for all $x \in W - M$ and all $t > 0$ ([2], Theorem 5). If we construct V' and U^* as above, it is clear that there are no recurrent orbits in $U^* - M$.

Conversely, let U be a neighborhood of M such that $(U - M) \cap \mathcal{R} = \emptyset$. Then $M^* = (M \cup \mathcal{R}) \cap U = M$, and, since M is Liapunov stable Lemma 7 implies that M is asymptotically stable.

The absence of generalized recurrent orbits in X may be regarded as an instability property of $\mathcal{F}$. As Theorem 2 shows, $\mathcal{R} = \Phi$ is equivalent to the existence of a continuous real valued function which is strictly decreasing along every orbit. As we shall see, this property actually lies between two known instability criteria.

The dynamical system $\mathcal{F}$ is said to be <u>parallelizable</u> if there exists a set $S \subset X$ which intersects every orbit of $\mathcal{F}$, and a homeomorphism h of X onto $S \times E$ such that $h(xt) = (x,t)$, whenever $x \in S$. The set S is called a <u>global</u> <u>section</u> for $\mathcal{F}$. In [1], Dugundji and Antosiewicz prove that $\mathcal{F}$ is parallelizable if and only if it is <u>dispersive</u> -- that is, if $x, y \in X$ there are neighborhoods U_1 and U_2 of x and y respectively, and a constant $T > 0$ such that $U_1 t \cap U_2 = \Phi$, for $|t| > T$. Now, this is the same thing as saying that $J_1(x) = \Phi$, for all $x \in X$, and it follows by a trivial induction that $J_\alpha(x) = \Phi$, for all $x \in X$. Then <u>a</u> <u>fortiori</u> $\mathcal{R} = \Phi$.

$\mathcal{F}$ is said to be <u>completely</u> <u>unstable</u> if all points are wandering. That is $x \notin J_1(x)$, for all $x \in X$, and $\mathcal{R}_1 = \Phi$. Therefore, if $\mathcal{R} = \Phi$, then $\mathcal{F}$ is certainly completely unstable.

<u>Theorem 6</u>. (i) <u>If</u> $\mathcal{F}$ <u>is</u> <u>parallelizable</u>, $\mathcal{R} = \Phi$.
(ii) <u>If</u> $\mathcal{R} = \Phi$, $\mathcal{F}$ <u>is</u> <u>completely</u> <u>unstable</u>.
(iii) <u>If</u> $\mathcal{F}$ <u>is</u> <u>completely</u> <u>unstable</u>, <u>and</u> <u>if</u> $D_1(x) = \gamma^+(x)$, for all $x \in X$, <u>then</u> $\mathcal{F}$ <u>is</u> <u>parallelizable</u>.

Proof. (i) and (ii) have already been proved. To prove (iii), we show that $\mathcal{F}$ is dispersive. Let $x \in X$ and suppose $y \in J_1(x)$. Then $y \in \gamma^+(x)$, so $y = x\tau$, for some $\tau > 0$. Therefore $x\tau \in J_1(x)$, and $x \in J_1(x)(-\tau) = J_1(x)$, which contradicts complete instability.

The converses of (i) and (ii) do not hold in general. Consider the dynamical system $\mathcal{F}$ defined by the system

$$\begin{cases} \frac{dx}{dt} = \sin y \\ \frac{dy}{dt} = \cos^2 y. \end{cases}$$

The orbits of $\mathcal{F}$ are the curves $x = c + \sec y$, and the lines $y = (2k + 1)\frac{\pi}{2}$ $(k = 0, \pm 1, \ldots)$.

Let $\mathcal{F}_o$ be the dynamical system obtained by restricting $\mathcal{F}$ to the strip $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. In this case, $\mathcal{R} = \Phi$, since the function $f(x, y) = -y(\arctan x + \pi)$ is strictly decreasing along every orbit. It is easy to see that $\mathcal{F}_o$ is not parallelizable; indeed, it has an improper saddle point ([5], p. 411).

To see that the converse of (ii) is false, consider the dynamical system $\mathcal{F}_1$ obtained from $\mathcal{F}$ by identifying (x, y) and $(x, y + 2\pi)$. $\mathcal{F}_1$ has the same local properties as $\mathcal{F}_o$, and is therefore completely unstable. For any real x, $(x, \frac{3\pi}{2}) \in J_1(x, \frac{\pi}{2})$ and $(x, \frac{\pi}{2}) \in J_1(x, \frac{-\pi}{2})$ so $(x, \frac{-\pi}{2}) = (x, \frac{3\pi}{2}) \in J_1^2(x, \frac{-\pi}{2}) \subset J_2(x, \frac{-\pi}{2})$ and therefore $\mathcal{R}_2 \neq \Phi$.

An interesting problem is to consider the consequences of the assumption $D_1(x) = \gamma^+(x)$, for all $x \in X$. If, in addition to this hypothesis, it is assumedthat there are no singular or periodic orbits, I conjecture that $\mathcal{F}$ is completely unstable (and therefore parallelizable).

5. <u>The proof of Theorem</u> 4. Let $X = \bigcup_{j=1,2,\ldots} K_j$, where each K_j is compact, and K_{j+1} is a neighborhood of K_j $(j = 1,2, \ldots)$.

Let Λ denote the dyadic rational numbers λ with $0 \leq \lambda < 1$. We shall define, for each $\lambda \in \Lambda$, a set U_λ such that 1) y is in every U_λ and x is in no U_λ, 2) If $\lambda, \lambda' \in \Lambda$, with $\lambda < \lambda'$ then $U_\lambda \subset$ interior $U_{\lambda'}$, 3) If $z \in U_\lambda$ and $z' \prec z$, then $z' \in U_\lambda$, and 4) $U_\lambda \cap K_j$ is compact $(\lambda \in \Lambda,\ j = 1,2, \ldots)$.

Once the sets U_λ with these properties are defined, we may define $f(z) = \inf\{\lambda \in \Lambda | z \in U_\lambda\}$, and $f(z) = 1$ if z is in no U_λ. It is easy to verify that f is continuous and has the necessary properties.

If $z \in X$ let $L(z) = \{z' \in X | z' \prec z\}$ and if $A \subset X$, let $L(A) = \bigcup_{z \in A} L(z)$. Since $\prec$ is transitive, $L(L(A)) = L(A)$.

By renumbering if necessary, we may suppose that x and y are in the interior of K_1. Let W_0 and N be disjoint compact neighborhoods of y and x respectively such that $L(W_0) \cap N = \Phi$. Such neighborhoods exist, since $\prec$ is closed. Let $U_{0,1} = L(W_0) \cap K_1$. Then $U_{0,1}$ is compact, $L(U_{0,1}) \cap K_1 = U_{0,1}$, and $U_{0,1} \cap N = \emptyset$. Now, let $W_{\frac{1}{2},1}$ be a compact neighborhood of $U_{0,1}$ in K_1 such that $L(W_{\frac{1}{2},1}) \cap N = \emptyset$. Let $U_{\frac{1}{2},1} = L(W_{\frac{1}{2},1}) \cap K_1$. Then $L(U_{\frac{1}{2},1}) \cap K_1 = U_{\frac{1}{2},1}$, and $U_{\frac{1}{2},1} \cap N = \emptyset$. Also $U_{0,1} \subset$ interior $U_{\frac{1}{2},1}$. Continuing in this manner, we construct $U_{3/4,1}$, $U_{7/8,1}$, ... To define, for example, $U_{1/4,1}$ proceed as follows. Let $N^* =$ closure $(K_1 - U_{\frac{1}{2},1})$. Then $U_{0,1} \cap N^* = \emptyset$, and we may let N^* play the role of N above. That is, we find a compact neighborhood $W_{1/4,1}$ of $U_{0,1}$ such that $L(W_{1/4,1}) \cap N^* = \emptyset$. This is possible since $L(U_{0,1}) \cap K_1 = U_{0,1}$. Then, let $U_{1/4,1} = L(W_{1/4,1}) \cap K_1$. Clearly, $U_{1/4,1} \subset$ interior $U_{\frac{1}{2},1}$.

By this process, we obtain compact sets $U_{\lambda,1}$ $(\lambda \in \Lambda)$ in K_1 such that $y \in$ interior $U_{\lambda,1}$, $x \notin U_{\lambda,1}$, and, if $\lambda < \lambda'$, then $U_{\lambda,1} \subset$ interior $U_{\lambda',1}$ ("interior" is relative to K_1).

Next we define $U_{\lambda,2}$ analogously using K_2 in place of K_1. We start with the same W_o and N, and define $U_{o,2} = L(W_o) \cap K_2 = L(U_{o,1}) \cap K_2$. Now, let $W_{\frac{1}{2},2}$ be a compact neighborhood of $U_{o,2}$ in K_2 such that $W_{\frac{1}{2},2} \cap K_1 = W_{\frac{1}{2},1}$, $L(W_{\frac{1}{2},2}) \cap N = \emptyset$, and $L(W_{\frac{1}{2},2}) \cap K_1 = U_{\frac{1}{2},1}$. That this is possible again follows from the "closed" property of $\prec$. Proceed as above, always requiring $W_{\lambda,2} \cap K_1 = W_{\lambda,1}$, $L(W_{\lambda,2}) \cap K_1 = U_{\lambda,1}$, and $L(W_{\lambda,2}) \cap N = \emptyset$. Define $U_{\lambda,2} = L(W_{\lambda,2}) \cap K_2$. Obviously $U_{\lambda,1} \subset U_{\lambda,2}$ for each $\lambda \in \Lambda$.

For each positive integer k, define $U_{\lambda,k}$ in this manner, and let $U_\lambda = \bigcup_{k=1,2,\ldots} U_{\lambda,k}$. Properties 1) and 2) follow from the corresponding properties of the $U_{\lambda,k}$, and 4) is true since $U_\lambda \cap K_j = U_{\lambda,j}$. Let $z \in U_\lambda$, and suppose $z' \prec z$. Then $z \in U_{\lambda,j}$, $z' \in K_\ell$ for some j and ℓ. If $m = \max(j, \ell)$, $z \in U_{\lambda,m}$, $z' \in K_m$ and $z' \in L(U_{\lambda,m}) \cap K_m = U_{\lambda,m} \subset U_\lambda$. Hence 3) is true.

RIAS and University of Maryland.

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